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Rational Combinations of x^{λ_k}, $\lambda_k \geqslant 0$ Are Always Dense in $C[0, 1]$

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Several years ago, the second author conjectured that the set of ratios of finite linear combinations of given, distinct monomials $x^{\lambda_1}, x^{\lambda_2},\dots$ is dense in $C[0, 1]$, assuming all λ_k are $\geqslant 0$. This rather bold conjecture was proven true by Somorjai [1] with the assumption that $\lambda_k \to \infty$ as $k \to \infty$. By an elementary argument, the same result can be extended to any sequence of distinct monomials as long as $\lambda_1, \lambda_2,\dots$ are positive and bounded away from zero. For if $\{\lambda_k\}_{k=1}^{\infty}$ has a positive limit point, the set of *linear* combinations of x^{λ_k} is dense in $C[0, 1]$. (See [2].) However, the above arguments fail if $\lambda_k \to 0$ as $k \to \infty$. The purpose of this note is to show that the original conjecture is true in all cases. Toward that end, we will prove:

THEOREM. *Suppose $\lambda_k > 0$, $k = 1, 2,\dots$, and $\lambda_k \to 0$ as $k \to \infty$. Then for any $f \in C[0, 1]$ and $\epsilon > 0$, there exists a rational function $R(x) = \sum_{k=1}^{M} \alpha_k x^{\lambda_k} / \sum_{k=1}^{M} \beta_k x^{\lambda_k}$ such that $\| f - R \|_{[0,1]} \leqslant \epsilon$.*

Proof. By making a routine change of variables, it clearly suffices to establish the result on a subinterval $[0, a]$, $a > 0$. In fact, we will show that any continuous function on $[0, 1/e]$ can be approximated to within a preassigned ϵ by the given rational functions. Also, by passing to a subsequence, we may assume without loss of generality that $\{\lambda_k\}_{k=1}^{\infty}$ is a strictly decreasing sequence. Suppose then that f is continuous on $[0, 1/e]$ and $\epsilon > 0$ is given. Then, if we set

$$g(u) = f(e^{-1/u}), \qquad 0 \leqslant u \leqslant 1$$

with the understood limit at $u = 0$, it follows that g is continuous on $[0, 1]$

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and by Weierstrass' theorem, there exists an Nth degree polynomial $\sum_{k=0}^{N} a_k u^k$ such that

$$\left\| g(u) - \sum_{k=0}^{N} a_k u^k \right\|_{[0,1]} \leqslant \frac{\epsilon}{2}.$$

Thus, writing $x = e^{-1/u}$,

$$\left\| f(x) - \sum_{k=0}^{N} a_k \left(\frac{-1}{\log x}\right)^k \right\|_{[0,1/e]} \leqslant \frac{\epsilon}{2}. \tag{1}$$

Now, let $\sum_{k=0}^{N} | a_k | = A$ and assume (again by passing to a subsequence, if necessary) that $\lambda_1 \leqslant \epsilon/2A$. To reapproximate the "polynomial" in (1), let $P_k(x)$ denote the kth divided difference of x^λ at $\lambda = \lambda_1, \lambda_2, ..., \lambda_{k+1}$ for $k = 0, 1, 2, ..., N - 1$, i.e.,

$$P_0(x) = P_0(x; \lambda_1) = x^{\lambda_1},$$

$$P_1(x) = P_1(x; \lambda_1, \lambda_2) = (x^{\lambda_1} - x^{\lambda_2})/(\lambda_1 - \lambda_2),$$

and, in general

$$P_k(x) = \frac{P_{k-1}(x; \lambda_1, ..., \lambda_k) - P_{k-1}(x; \lambda_2, ..., \lambda_{k+1})}{\lambda_1 - \lambda_{k+1}}, \qquad k = 1, 2, ..., N - 1,$$

Then [3, p. 210]

$$P_k(x) = \frac{x^{h_k}(\log x)^k}{k!}, \qquad \lambda_N \leqslant h_k \leqslant \lambda_1; \quad k = 0, 1, 2, ..., N - 1.$$

Also, let $P_N(x) = P_N(x; \lambda_N, \lambda_{N+1}, ..., \lambda_{2N})$ denote the Nth divided difference based at the indicated points so that

$$P_N(x) = \frac{x^{h_N}(\log x)^N}{N!}, \qquad \lambda_{2N} \leqslant h_N \leqslant \lambda_N.$$

If we then set

$$R(x) = \frac{\sum_{k=0}^{N} (-1)^k\, k!\, a_k P_{N-k}(x)}{N!\, P_N(x)}$$

it follows that R is a rational combination of the monomials x^{λ_k}, and

$$R(x) = \sum_{k=0}^{N} a_k \left(\frac{-1}{\log x}\right)^k x^{\eta_k},$$

$$\text{with} \quad \eta_0 = 0, \quad 0 < \eta_k \leqslant \lambda_1, \quad k = 1, 2, ..., N.$$

Hence

$$\left\| \sum_{k=0}^{N} a_k \left(\frac{-1}{\log x}\right)^k - R(x) \right\|_{[0,1/e]} \leqslant \sum_{k=1}^{N} |a_k| \max_{1 \leqslant k \leqslant N} \left\| \frac{1 - x^{\eta_k}}{(\log x)^k} \right\|_{[0,1/e]}. \quad (2)$$

Note, however, that for $\eta > 0$, $(1 - x^{\eta})|\log x|$ is a positive increasing function on (0, 1) with $\lim_{x \to 1-} (1 - x^{\eta})/|\log x| = \eta$ so that

$$\left\| \frac{1 - x^{n}}{\log x} \right\|_{[0,1/e]} \leqslant \eta.$$

Also, for $0 \leqslant x \leqslant 1/e$, $|\log x| > 1$ so that

$$\left\| \frac{1 - x^{\eta_k}}{(\log x)^k} \right\|_{[0,1/e]} \leqslant \eta_k \leqslant \lambda_1 .$$

Thus,

$$\max_{1 \leqslant k \leqslant N} \left\| \frac{1 - x^{\eta_k}}{(\log x)^k} \right\|_{[0,1/e]} \leqslant \lambda_1 \leqslant \frac{\epsilon}{2A}$$

and by (1) and (2) the proof is complete.

Note. While the approximating rational function was of the form

$$R(x) = \frac{\alpha_{2N} x^{\lambda_{2N}} + \cdots + \alpha_1 x^{\lambda_1}}{\beta_{2N} x^{\lambda_{2N}} + \cdots + \beta_1 x^{\lambda_1}},$$

i.e., $R = P/Q$ with $P(0) = Q(0) = 0$, the construction can easily be modified to obtain an approximating rational function with nonzero denominator. Indeed, since $Q(x) \neq 0$ for $x > 0$ we may assume $Q(x) > 0$ and then we need only set

$$R^*(x) = \frac{\gamma\delta + P}{\delta + Q}, \qquad \text{with} \quad \gamma = \frac{\alpha_{2N}}{\beta_{2N}}$$

and sufficiently small $\delta > 0$.

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